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Stochastic Particle Acceleration

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The propagation of test particles in a turbulent plasma is considered both analytically and numerically. We split the Green's function into a deterministic and a stochastic part and derive several exact relations between them; these relations show that partial summations of the perturbation series do not lead, in general, to a consistent closure for the hierarchy problem. The mean propagator in velocity space is calculated numerically for an acceleration field given by an Ornstein–Uhlenbeck process. If the relative coherent change ϵ of the particle velocity remains smaller than 0.1% the Fokker–Planck description may be adequate for the time evolution of the propagator. Otherwise the picture of Brownian motion becomes worse, and incorporating memory effects or renormalization of the mass operator have also been found to disagree with the numerical experiment.

KEY WORDS: Stochastic acceleration; Green's function for stochastic equations; plasma turbulence; numerical simulation.

1. INTRODUCTION

A basic problem in plasma physics is the necessity of solving the collisionless Boltzmann equation (Vlasov's equation) for the phase space density of particles which are exposed to a random accelerating field $\mathbf{b} = (e/m)[\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]$; here e, m are the particle's charge and mass, respectively, $\mathbf{v}$ its velocity, and $\mathbf{E}$, $\mathbf{B}$ are the electric and magnetic fields which may consist of deterministic or equilibrium fields (external and/or internal) plus a random part due to some plasma instability. In the numerical part of this paper we address the problem where only a random field is present. Even if it is assumed to be known in a statistical sense there remains the closure problem which is common to all differential equations with stochastic coefficients. The usual "weak-coupling limit" leading to a consistent picture

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of Brownian motion can sometimes be justified (a review of "intrinsic" closures has been given earlier[3]), but in practice these conditions are not always met. So the question arises whether we have the proper perspectives for a really better theory. The answer is "no," as will be shown in the present paper.

In Section 2 we formulate the problem in terms of averaged and stochastic Green's functions, $\bar{G}$ and $\tilde{G}$. The formal procedure of eliminating $\tilde{G}$ can be viewed as a projection technique[4] (if the ensemble average $\langle \tilde{G} \rangle$ is chosen to be zero); this and similar methods have been described and used many times in connection with stochastic equations, e.g., in Refs. 5–7. The formulation of the present problem in terms of Green's functions has also been given several times, [2,8–15] but it seems that not all answers would be "no" to the question as stated above. Here we give some exact relations between $\bar{G}$ and $\tilde{G}$ which show that—in the case of Gaussian statistics of $\mathbf{b}$—the lowest-order truncation for $\tilde{G}$ is only consistent if $\bar{G}$ is nearly the same as the free propagator, a result which could also be anticipated generally by intuition.

In the following sections we describe a one-dimensional numerical experiment: We calculated the trajectories of several hundred thousands of particles in a stochastic force field $b(x,t)$ with mean and two-point correlation given by

$$\langle b(x,t) \rangle = 0$$

$$\langle b(x,t)b(x',t') \rangle = b_0^2 \exp\left(- \frac{|x-x'|}{l_c} \right) \exp\left(- \frac{|t-t'|}{\tau_c} \right)$$

We then obtained the "exact" mean propagator $\langle G \rangle$ from the trajectories and used this information to test the validity of approximate equations for $\langle G \rangle$: The Fokker–Planck equation for the space-averaged part is found to be a satisfactory approximation for short correlation times τ_c and small relative coherent change ϵ of the particle velocity ($\lesssim 0.1\%$); for larger τ_c or $\epsilon \gtrsim 0.1\%$ this situation is changed, and the discrepancy increasing with ϵ can *not* be diminished by including a memory effect as proposed for other stochastic equations[1] or by renormalizing the lowest-order term of the mass operator,[2] in agreement with the result of Section 2. Finally, we conclude that the Fokker–Planck description of anomalous plasma diffusion by modes of low frequency (large τ_c) is only valid if the plasma diffusion is more than two orders of magnitude smaller than corresponding to the Bohm diffusion formula.[16]

2. EXACT PROPAGATOR RELATIONS

We consider particles exposed to a random acceleration field $\mathbf{b} = \mathbf{b}(\mathbf{x}, \mathbf{v}, t)$; the particle density in μ-space, $f = f(\mathbf{x}, \mathbf{v}, t)$, is then a solution

of Vlasov's equation:

$$Lf = 0 \tag{1}$$

$$L = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{v}} \tag{2}$$

The solution evolving from an initial smooth distribution $f_0(\mathbf{x}, \mathbf{v})$ is conveniently obtained from the retarded Green's function of Eq. (1), $G = G(\mathbf{x}, \mathbf{v}, t;\ \mathbf{x}', \mathbf{v}', t')$:

$$f(\mathbf{x}, \mathbf{v}, t) = \int d^3x' \int d^3\mathbf{v}' G(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', 0) f_0(\mathbf{x}', \mathbf{v}') \tag{3}$$

with

$$LG = \delta^3(\mathbf{x} - \mathbf{x}')\delta^3(\mathbf{v} - \mathbf{v}')\delta(t - t') \tag{4}$$

$$G = 0 \qquad \text{for} \quad t < t' \tag{5}$$

Average properties of a plasma can be expressed by the ensemble average of f, $\langle f \rangle$; if the plasma is initially unstable we anticipate that the fluctuations of f evolve to a level which is several orders of magnitude higher than the thermal noise. Then $\langle f \rangle$ can be obtained from Eq. (3) after sufficiently long time by neglecting the initial correlations in x-space, i.e., by replacing $f_0(\mathbf{x}', \mathbf{v}') \to \langle f_0(\mathbf{v}') \rangle$ and $G \to \langle G \rangle$. It is, therefore, of interest to obtain an equation for the mean propagator $\langle G \rangle$. If $\overline{G}$ denotes *any* deterministic propagator derived from G, with the same initial conditions as G, one can introduce the splitting of G as follows:

$$G = \overline{G} + \tilde{G} \tag{6}$$

and eliminate the stochastic part $\tilde{G}$ in an exact but formal way. Here $\langle \tilde{G} \rangle$ can be chosen freely, with the only restriction that its value for $t = t'$ is zero; if we choose $\langle \tilde{G} \rangle = 0$, the definition $\overline{G} = \langle G \rangle$ results. In this section we derive the formal equation for $\overline{G}$ ("Dyson equation") and some exact relations between $\tilde{G}$ and $\overline{G}$.

Let us denote the ensemble average of L by $\mathring{L}$:

$$\mathring{L} = \langle L \rangle = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \langle \mathbf{b} \rangle \cdot \frac{\partial}{\partial \mathbf{v}} \tag{7}$$

where $\langle \mathbf{b} \rangle$ may be due to a static magnetic field and/or to an externally applied oscillating electric field which drives the instability. Equation (4) can then be rewritten in a shorthand notation as follows:

$$(\mathring{L} + \delta L)\overline{G} + L\tilde{G} = 1 \tag{8}$$

where 1 is an abbreviation of the product of δ functions, and

$$\delta L = \delta \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{v}} \tag{9}$$

$$\delta \mathbf{b} = \mathbf{b} - \langle \mathbf{b} \rangle \tag{10}$$

The average of Eq. (8) is the "Dyson equation" for $\bar{G}$:

$$(\mathring{L} + M)\bar{G} = 1 \tag{11}$$

where the mass operator M is defined by

$$M\bar{G} = \langle L\tilde{G} \rangle \tag{12}$$

M has to be understood as an integral operator acting on the first set of variables in $\bar{G}$, $\mathbf{x}, \mathbf{v}, t$. In order to eliminate $\tilde{G}$ in favor of M we define some new quantities:

$$\bar{L} = \mathring{L} + M \tag{13}$$

$$\tilde{L} = L - \bar{L} = \delta L - M \tag{14}$$

Then our basic equation (8) can be rewritten and formally integrated in two equivalent forms:

$$\tilde{L}\bar{G} + L\tilde{G} = 0 \tag{15a}$$

$$\tilde{L}(\bar{G} + \tilde{G}) + \bar{L}\tilde{G} = 0 \tag{15b}$$

$$\tilde{G} = -(\bar{G} + \tilde{G})\tilde{L}\bar{G} \tag{16a}$$

$$\tilde{G} = -\bar{G}\tilde{L}(\bar{G} + \tilde{G}) \tag{16b}$$

where again all quantities are understood as integral operators. Equations (16a and 16b) can be solved for $\tilde{G}$ by a von Neumann series with the following result:

$$\tilde{G} = \bar{G} \sum_{n=1}^{\infty} (-\tilde{L}\bar{G})^n \tag{17a}$$

$$= \sum_{n=1}^{\infty} (-\bar{G}\tilde{L})^n \bar{G} \tag{17b}$$

The equivalence of Eqs. (17a) and (17b) is obvious, but the equivalence of Eqs. (16a) and (16b) is less obvious and implies the following "commutator rule" for $\tilde{G}$:

$$\tilde{G}\tilde{L}\bar{G} = \bar{G}\tilde{L}\tilde{G} \tag{18}$$

A second commutator rule can be obtained as follows: Integrating the identity $\bar{L}\tilde{G} \equiv \bar{L}\tilde{G}$ we obtain

$$\tilde{G} = \bar{G}\bar{L}\tilde{G} \tag{19}$$

Because of Eqs. (11) and (13) we have also

$$\bar{L}\bar{G} = 1 \tag{20}$$

which may be inserted into Eq. (19) to give

$$\tilde{G}\bar{L}\bar{G} = \bar{G}\bar{L}\tilde{G} \tag{21}$$

Adding Eqs. (18) and (21) we obtain our second commutator rule:

$$\tilde{G}L\bar{G} = \bar{G}L\tilde{G} \tag{22}$$

Finally we obtain an implicit equation for the mass operator from Eqs. (12) and (17b):

$$M = \mathring{L}\langle\tilde{G}\rangle\bar{L} + \sum_{n=1}^{\infty}\left\langle \delta L(-\bar{G}\tilde{L})^n \right\rangle \tag{23}$$

where $\langle\tilde{G}\rangle$ can be freely chosen, as stated above. Choosing now $\langle\tilde{G}\rangle = 0$ and retaining in Eq. (23) only the lowest term $n = 1$ leads to an expression for M which is formally of second order in δL, namely,

$$M \approx M^{(2)} = -\langle \delta L\bar{G}\delta L\rangle \tag{24}$$

But the series in Eqs. (17a) and (17b) and (23) are not ordinary perturbation expansions for "small" δL, and one could suppose that Eq. (24) is a good approximation for M even in cases where $\bar{G}$ differs strongly from the "free" propagator $\mathring{G}$, defined according to the following equation:

$$\mathring{L}\mathring{G} = 1 \tag{25}$$

Indeed a Dyson equation with $M \equiv M^{(2)}$ is the exact result for the Kraichnan model where the closure is *not* due to a smallness parameter.[2] Evaluating Eq. (24) for our present problem leads to the following equation for $\bar{G}$, written in full length:

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \langle\mathbf{b}\rangle\cdot\frac{\partial}{\partial\mathbf{v}}\right]\bar{G}(\mathbf{x},\mathbf{v},t;\mathbf{x}',\mathbf{v}',t') \\
&\quad - \frac{\partial}{\partial\mathbf{v}}\cdot\int d^3x''\int d^3\mathbf{v}''\int dt''\,\langle\delta\mathbf{b}(\mathbf{x},\mathbf{v},t)\delta\mathbf{b}(\mathbf{x}'',\mathbf{v}'',t'')\rangle \\
&\quad \times \bar{G}(\mathbf{x},\mathbf{v},t;\mathbf{x}'',\mathbf{v}'',t'')\cdot\frac{\partial}{\partial\mathbf{v}''}\bar{G}(\mathbf{x}'',\mathbf{v}'',t'',\mathbf{x}',\mathbf{v}',t') \\
&= \delta^3(\mathbf{x}-\mathbf{x}')\delta^3(\mathbf{v}-\mathbf{v}')\delta(t-t')
\end{aligned} \tag{26}$$

Here we have used the fact that

$$\mathbf{b}\cdot\frac{\partial}{\partial\mathbf{v}} = \frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{b} \tag{27}$$

due to the particular form of the Lorentz force. Unfortunately Eq. (26) is nonlinear in $\bar{G}$, and solutions have only been obtained in two very restrictive cases.[2] It is, nevertheless, of interest to know the range of validity of Eq. (26) and of simpler versions. A simpler version is obtained if $\bar{G}$ in Eq.

(24) or $\bar{G}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}'', \mathbf{v}'', t'')$ in Eq. (26) is replaced by $\mathring{G}$; this "Bourret approximation" has already been proposed and studied in the context of other stochastic equations.[1] Finally we consider the space-averaged part of $\bar{G}$ in the case $\langle \mathbf{b} \rangle = 0$ and for spatially homogeneous statistics of $\delta\mathbf{b} = \delta\mathbf{b}(\mathbf{x}, t)$:

$$g(\mathbf{v}, t; \mathbf{v}') = \int d^3x \langle G(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', 0) \rangle \tag{28}$$

Then a "small" mass operator $M^{(2)}$ means a slow time dependence of g, and the memory effect of the Bourret approximation can be neglected; the result is then for times sufficiently larger than the correlation time, the ordinary Fokker–Planck equation for the Brownian motion of particles:

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathsf{D}(\mathbf{v}) \cdot \frac{\partial}{\partial \mathbf{v}} g \qquad \text{for} \quad t > 0 \tag{29}$$

$$\mathsf{D}(\mathbf{v}) = \int_0^\infty dt'' \langle \delta\mathbf{b}(\mathbf{v}t'', t'') \delta\mathbf{b}(0, 0) \rangle \tag{30}$$

It is the aim of this section to show for Gaussian statistics of $\delta\mathbf{b}$ that the lowest order of the expansion in Eqs. (17a) and (17b) is only relevant if $\bar{G} \approx \mathring{G}$. For this purpose we derive now an exact relation between M and $M^{(2)}$. We multiply Eq. (4) from the right by δL and take the average; the result is

$$\langle L G \delta L \rangle = 0 \tag{31}$$

Using $L = \mathring{L} + \delta L$ and $G = \bar{G} + \tilde{G}$ gives

$$\mathring{L} \langle \tilde{G} \delta L \rangle + \langle \delta L \bar{G} \delta L \rangle + \langle \delta L \tilde{G} \delta L \rangle = 0 \tag{32}$$

We add here a term $\mathring{L} \langle \tilde{G} \rangle \mathring{L}$ on both sides and obtain an equation for $\mathring{L} \langle \tilde{G} L \rangle$; eliminating $\langle \tilde{G} L \rangle$ by the average of Eq. (22) leads to the following mass operator formula:

$$\mathring{L} \bar{G} M = M^{(2)} - \langle \delta L \tilde{G} \delta L \rangle + \mathring{L} \langle \tilde{G} \rangle \mathring{L} \tag{33}$$

This exact formula is useful for testing approximations for $\tilde{G}$. Using the centered part of the lowest-order term,

$$\tilde{G} \approx \tilde{G}^{(1)} = -\bar{G} \delta L \bar{G} \tag{34}$$

we obtain for Gaussian statistics of $\delta\mathbf{b}$ from Eq. (33)

$$\mathring{L} \bar{G} M^{(2)} \approx M^{(2)} \tag{35}$$

and, therefore,

$$\bar{G} \approx \mathring{G} \tag{36}$$

It should be noted, however, that the Kraichnan model itself does not suffer from this inconsistency since it does not use Eq. (34). But it seems

rather plausible that in general $G \approx \bar{G}$ is only possible if also $\bar{G} \approx \mathring{G}$. In the following sections we confirm this conclusion also for a non-Gaussian example, and we find the parameter range where all theoretical models discussed above become invalid.

3. A METHOD TO TEST CLOSURE APPROXIMATIONS NUMERICALLY

In this section we describe the method used to test numerically Kraichnan's closure approximation for the stochastic acceleration problem [Eq. (26)]. We also applied it to the Bourret and Fokker–Planck equation, resulting as simplifications of (26), and compared the ranges of validity of the three theories. Our aim is to gather some empirical material telling us in which parameter range the Bourret, or respectively, Kraichnan, equation improves the Fokker–Planck description which is known to be valid for small amplitudes and short correlation times of the acceleration field.

Our computations are restricted to the one-dimensional case, and we assume homogeneity and stationarity of the acceleration field ensemble.

3.1. Basic Equations

The equations for the average propagator are given by

$$\frac{\partial}{\partial t} g(v,t;v') = \frac{\partial}{\partial v} D(v) \frac{\partial}{\partial v} g(v,t;v') \tag{37}$$

for the Fokker–Planck description,

$$\frac{\partial}{\partial t} g(v,t;v') = \frac{\partial}{\partial v} \int_0^t dt'' \langle bb \rangle (v(t-t''), t-t'') \times \frac{\partial}{\partial v} g(v,t'';v') \tag{38}$$

for Bourret's approximation, and

$$\frac{\partial}{\partial t} g(v,t;v') = \frac{\partial}{\partial v} \int_0^t dt'' \int_{-\infty}^{+\infty} dv'' \int_{-\infty}^{+\infty} dx'' \langle bb \rangle (x'', t-t'') \times g(0,v,t-t'';-x'',v'') \frac{\partial}{\partial v''} g(v'',t'';v') \tag{39}$$

for Kraichnan's theory. Here $g(v,t;\, v')$ and $D(v)$ are as in (28) and (30), respectively, and $g(x,v,t;\, x',v') := \langle G(x,v,t;\, x',v',t'=0) \rangle$. To write the equations in this simple form we assumed that

$$\langle b(x,t) \rangle = 0 \tag{40}$$

and defined

$$\langle bb \rangle (|\xi|, |\tau|) := \langle b(x+\xi, t+\tau) b(x,t) \rangle \tag{41}$$

$g(v,t;v')\,dv$ gives the probability that a particle, having velocity v' at $t=0$ and being accelerated by the stochastic field $b(x,t)$, will have a velocity between v and $v+dv$ at time t. Correspondingly, $g(x,v,t;x',v')\,dx\,dv$ gives the probability that a particle with coordinates (x',v') at $t=0$ will be in the μ-space interval $[x,x+dx]\times[v,v+dv]$ at time t.

The version of the equations which fits our numerical purposes best is arrived at by integrating Eqs. (37)–(39) over a small but finite velocity interval $[v_L,v_R]$:

$$\frac{\partial}{\partial t}\int_{v_L}^{v_R} dv\; g(v,t;v') = D(v_R)\,\frac{\partial}{\partial v}\,g(v,t;v')\big|_{v=v_R} - D(v_L)\,\frac{\partial}{\partial v}\,g(v,t;v')\big|_{v=v_L} \tag{42}$$

$$\frac{\partial}{\partial t}\int_{v_L}^{v_R} dv\; g(v,t;v') = \int_0^t dt''\left\{\langle bb\rangle(v_R(t-t''),t-t'')\,\frac{\partial}{\partial v}\,g(v,t'';v')\big|_{v=v_R} - \langle bb\rangle(v_L(t-t''),t-t'')\times\frac{\partial}{\partial v}\,g(v,t'';v')\big|_{v=v_L}\right\} \tag{43}$$

$$\frac{\partial}{\partial t}\int_{v_L}^{v_R} dv\; g(v,t;v') = \int_0^t dt''\int_{-\infty}^{+\infty} dv''\int_{-\infty}^{+\infty} dx'' \times\{\,g(0,v_R,t-t'';-x'',v'') - g(0,v_L,t-t'';-x'',v'')\}\times\langle bb\rangle(x'',t-t'')\,\frac{\partial}{\partial v''}\,g(v'',t'';v') \tag{44}$$

Equations (42)–(44) are easier to handle in numerical computations because a differentiation is replaced by a difference. For $|v_R-v_L|$ sufficiently small the results derived for this set of equations will also apply to (37)–(39).

3.2. The Basic Test Idea

We now describe the method applied to check the validity of Eqs. (37)–(39). The basic idea is as follows. Each of the equations poses an initial-value problem: The time evolution of the average propagator has to be determined by solving the respective equation under the assumption that, at $t=0$,

$$g(v,t=0;v') = \delta(v-v') \tag{45}$$

for (37) and (38), respectively, that

$$g(x, v, t = 0; x', v') = \delta(x - x')\delta(v - v') \tag{46}$$

for (39). Thus, if we already know the time evolution of $g(v, t; v')$ and $g(x, v, t; x', v')$ from numerical computations, then the criterion for the validity of any of the equations (37)–(39) is simply that it is fulfilled at every time when we insert $g(v, t; v')$, respectively, $g(x, v, t; x', v')$ into it. Our test method accordingly consists of the following steps:

(a) Find $g(v, t; v')$ for v' fixed, v, t variable, and $g(0, v, t; x', v')$ for v fixed, x', v', t variable.
(b) Use these quantities to form the (identical) left-hand sides of Eqs. (42)–(44).
(c) Compute the right-hand sides of (42)–(44).
(d) Compare the time evolution of these four quantities.

Steps (b) and (c) are performed on Eqs. (42)–(44) instead of (37)–(39) because the relevant quantities are more easily computed.

That equation whose right-hand side agrees most exactly with $(\partial/\partial t)$ $g(v, t; v')$ [or with $(\partial/\partial t)\int_{v_L}^{v_R} dv\ g(v, t; v')$, for that matter] we regard as the best approximate expression for the average propagator g. We should, of course, repeat this procedure for many different values of v and v' to completely determine where and when the equations are adequate. Computer costs set an upper limit to this endeavor.

3.3. Computation of the Average Propagator

So we first must compute $g(v, t; v')$ as a function of v and t, and $g(0, v, t; x', v')$ as a function of t, x', v'. According to the interpretation given in 3.1 this can be done by employing the following technique: By solving numerically Newton's equations of motion,

$$\begin{aligned} \dot{X}(t) &= V(t), \qquad && X(0) \text{ arbitrary} \\ \dot{V}(t) &= b(X(t), t), \qquad && V(0) = v' \end{aligned} \tag{47}$$

we find the trajectory of a particle with initial velocity v' for a large number ($\approx 10^5$) of different acceleration fields $b(x, t)$. They are constructed in such a way that they fulfill

$$\begin{aligned} \langle b(x,t)\rangle &= 0 \\ \langle b(x,t)b(x',t')\rangle &= b_0^2 \exp\left(-\frac{|x-x'|}{l_c}\right)\exp\left(-\frac{|t-t'|}{\tau_c}\right) \end{aligned} \tag{48}$$

The technique for computing the fields on the computer is described in the Appendix.

From the trajectories of the particle $g(v,t;v')$ can be found by subdividing the v-axis into a large number of cells with length Δv and counting how many trajectories lie in each of them at time t. If this number is equal to $N_v(t)$ for a cell centered around v then $g(v,t;v')$ is approximately given by

$$g(v,t;v') = \frac{N_v(t)}{N\Delta v} \tag{49}$$

where N is the total number of trajectories [number of realizations of $b(x,t)$].

A similar approach is used in particle simulations of plasmas to compute the charge density. Sophistications developed for that problem (like the PIC[17] method) could also be applied here.

In principle one can determine $g(0,v,t;x',v')$ as a function of t, x', and v' in much the same manner: Compute a large number of trajectories for sufficiently many initial values x', v', then count how many of them lie in a two-dimensional interval of area $\Delta x \Delta v$ centered in μ-space around $(0,v)$ (remember that v here has to be considered as fixed). If at time t that number is equal to $N_{x'v'}(t)$ then

$$g(0,v,t;x',v') = \frac{N_{x'v'}(t)}{N\Delta x \Delta v} \tag{50}$$

Practically, however, this method is prohibitively uneconomic: All the trajectories which are not in the interval around $(0,v)$ are in a way superfluous. We find their exact positions although we only need the information that they are outside our interval.

Fortunately, it is possible to compute $g(0,v,t;x',v')$ in a more elegant, less wasteful way. As mentioned in 3.1,

$$g(x,v,t;x',v')dx\,dv = \begin{array}{l}\text{probability that a particle with initial coordinates } (x', \\ v') \text{ will be found in the interval } [x, x+dx] \times [v, \\ v+dv] \text{ at time } t\end{array}$$

This, however, is not the only interpretation possible. We also have

$$g(x,v,t;x',v')dx'\,dv' = \begin{array}{l}\text{probability that a particle with coordinates } (x,v) \text{ at} \\ \text{time } t \text{ started initially in the} \\ \text{interval } [x', x'+dx'] \times [v', \\ v'+dv']\end{array}$$

This quantity now can be computed very easily: If we reverse the direction of time, Newton's equations (47) will tell us where a particle with given

coordinates originally started at time $t = 0$. Thus, to find $g(0, v, t; x', v')$ we solve numerically equations (47) backwards in time for a particle with coordinates $(0, v)$ at time t, stopping at $t = 0$. We repeat this computation for a large number of acceleration fields $b(x, t)$ with statistical properties given by (48). Then we count how many trajectories lie in each cell $[x', x' + dx'] \times [v', v' + dv']$ of the (x', v') plane at $t = 0$, and finally find $g(0, v, t; x', v')$ by dividing this number by the cell area $\Delta x' \Delta v'$ and the total number of trajectories.

An additional simplification results from the stationarity of the b ensemble: Retracing the trajectories not from t to 0 but from t to $t_0 > 0$ will give us $g(0, v, t - t_0; x', v')$.

Let us summarize our strategy for finding $g(v, t; v')$ and $g(0, v, t; x', v')$: To compute $g(v, t; v')$ we solve Newton's equations as an *initial-value problem*) for very many different realizations of the acceleration field $b(x, t)$. To find $g(0, v, t; x', v')$ we solve Newton's equations as a *final-value problem* for a large number of b's. Parallel to the forward-in-time evolution of the first and the backward-in-time evolution of the second trajectory ensemble we compute $g(v, t; v')$ and $g(0, v, t; x', v')$ as described above.

4. PROGRAM DESCRIPTION AND TESTING

We briefly list the different parts of the program and the numerical techniques used.

First we have to compute a large number of acceleration fields obeying (48) (cf. Appendix). A method for generating normally distributed random numbers is described in Ref. 18. In addition most machines offer a subroutine library containing a Gaussian random number generator. With its aid the construction of the piecewise constant fields described in the Appendix is straightforward.

Then Eqs. (47) are used to compute the *forward* particle trajectory for every acceleration field: Time *increases* from $t = 0$ to $t = t_E$, *initial* values at $t = 0$ are $v = v'$, x arbitrary because the b ensemble is homogeneous. The solution of the equations was obtained by the leap-frog method. In the same way the *backward* particle trajectory was computed for every acceleration field and for two different *final* velocities [cf. (44)]. Time now *decreases* from $t = t_E$ to $t = 0$, *final* values at $t = t_E$ are $v = v_R$, $x = 0$, and $v = v_L$, $x = 0$, respectively. Reversing the sign of the time step makes the leap-frog algorithm run backwards in time.

The next part consists of determining $g(v, t; v')$ from the *forward*, and $g(0, v_L, t; x', v')$ as well as $g(0, v_R, t; x', v')$ from the *backward* trajectory ensembles. $g(v, t; v')$ is given by formula (49), and

$$g(0, v_L(v_R), t; x', v') = \frac{N_{x'v'}(t, v_L(v_R))}{N \Delta x' \Delta v'} \tag{51}$$

where $N_{x'v'}$ equals the number of trajectories lying in a cell centered around (x', v') with area $\Delta x' \Delta v'$.

Finally the left- and right-hand sides of Eqs. (42)–(44) are formed and plotted as a function of time. Derivatives are evaluated according to

$$\frac{df(x)}{dx} \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \tag{52}$$

Integrations are done by the trapezoid rule. The correlation function $\langle bb \rangle(x, t)$ occurring in the integrands was computed explicitly by summing over all acceleration fields. We did not use formula (48).

The program was thoroughly tested to make sure that it performs correctly. We first verified that the acceleration field ensemble is homogeneous and stationary. In Fig. 1 we show the behavior of the correlation function under a shift in x and t. Since $b(x, t)$ is constructed on a grid in (x, t) space $\langle bb \rangle(\xi, \tau) \equiv \langle b(x + \xi, t + \tau) b(x, t) \rangle$ is a step function, as is demonstrated in Fig. 1a. The continuous curves result from connecting the step centers and agree with the theoretical behavior given by (48).

The next question arising is whether the finite step size $(\Delta x, \Delta t)$ of the piecewise constant function $b(x, t)$ influences the particle behavior. Since we want to simulate the limit in which the correlation function is exactly given by (48) we must choose Δx and Δt sufficiently small. Figures 2 and 3 show what happens when we change the step lengths. The right- and

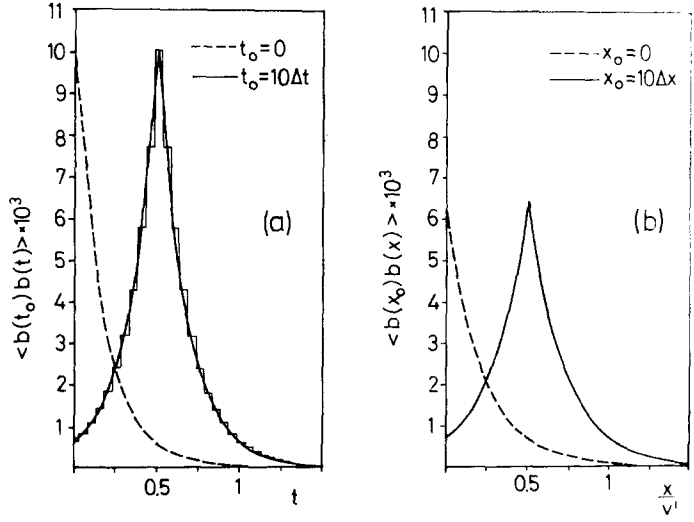

Fig. 1. Numerically computed correlation functions. The step function in (a) shows the actual shape; continuous lines result from connecting the step centers.

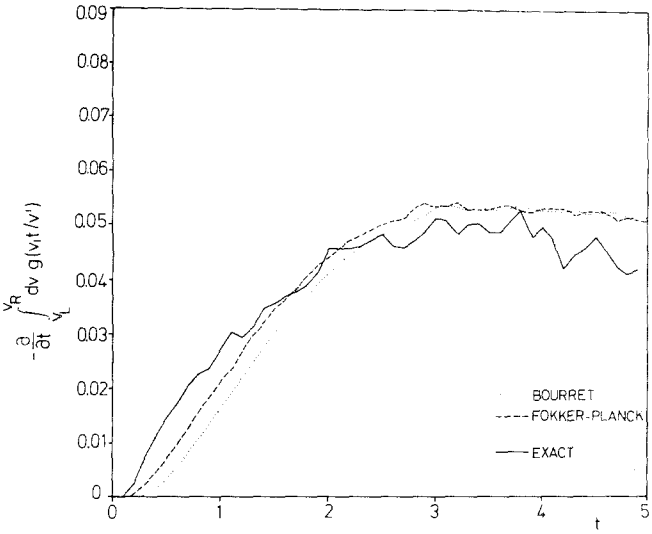

Fig. 2. Right- and left-hand sides of Eqs. (42) and (43). Step sizes for $b(x, t)$ are $\Delta x = 0.1$, $\Delta t = 0.1$.

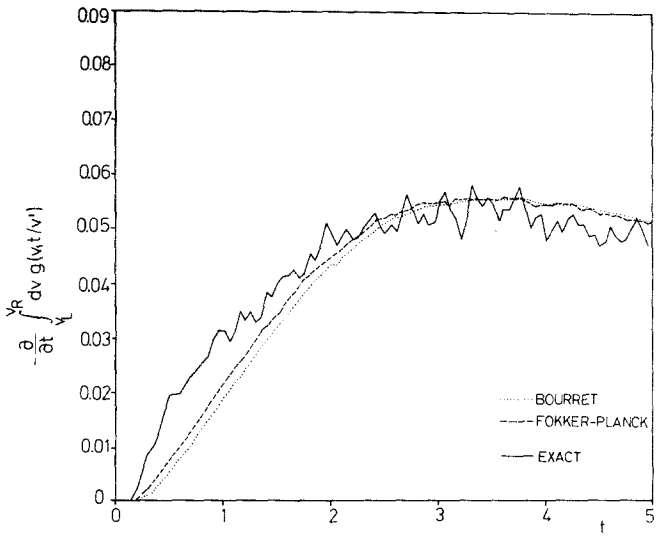

Fig. 3. Right- and left-hand sides of Eqs. (42) and (43). Step sizes for $b(x, t)$ are $\Delta x = 0.05$, $\Delta t = 0.05$.

left-hand sides of (42) and (43) are plotted as a function of time for two different choices of Δx and Δt: $\Delta x = 0.1$, $\Delta t = 0.1$ (Fig. 2) and $\Delta x = 0.05$, $\Delta t = 0.05$ (Fig. 3). The agreement is satisfactory.

Finally we examined if the propagators are computed exactly from the particle trajectories, and if the left- and right-hand sides of Eqs. (42)–(44) are computed correctly from the propagators. We restricted this test to equation (43). The right-hand side of (44) can then be checked by putting $b(x,t)$ equal to zero in the computation of $g(0, v_L(v_R), t; x', v')$. Equations (43) and (44) must be identical for this special case. A similar argument holds for (42): It must become identical to (43) if the correlation function falls off rapidly enough. To test the evaluation of the right-hand side of (43) we made the program skip the computation of the trajectory, i.e., the particle keeps its initial position and velocity all the time. The statistical element is introduced by choosing a Maxwell distribution with mean v' for the initial particle velocity (instead of $v = v'$ for every realization). We will then have

$$g(v,t;v') = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{1}{2} \frac{(v-v')^2}{\sigma^2} \right] \tag{53}$$

and, using (48), the right-hand side of (43) can be evaluated analytically. Table I shows how the result compares with the program output. The left-hand side of (43) should be zero for this special case, which is what the program computes. We also compared the values of $\int_{v_L}^{v_R} dv\, g(v,t;v')$ and

Table I. Right-Hand Side of (43), Evaluated Analytically by Using (53) and (48), and Computed Numerically.[a]

	Right-hand side of (43)	
t	Theory	Computation
$1\,\Delta t$	− 0.082	− 0.081
$2\,\Delta t$	− 0.231	− 0.229
$3\,\Delta t$	− 0.361	− 0.359
$4\,\Delta t$	− 0.476	− 0.473
$5\,\Delta t$	− 0.575	− 0.572
$6\,\Delta t$	− 0.662	− 0.658
$7\,\Delta t$	− 0.738	− 0.733
$8\,\Delta t$	− 0.804	− 0.799
$9\,\Delta t$	− 0.863	− 0.856

[a] Parameters are $\sigma = 0.02$, $v' = 2.00$, $b_0^2 = 2.53 \times 10^{-3}$, $l_c = 1.25$, $\tau_c = 1.0$, $v_L = 1.98$, $v_R = 2.02$.

found for $v_L = 1.98$, $v' = 2.00$, $v_R = 2.02$, $\sigma = 0.02$

$$\int_{v_L}^{v_R} dv\ g(v,t;v') = \begin{cases} 0.6826 & \text{analytically} \\ 0.6830 & \text{from the program} \end{cases} \tag{54}$$

We conclude that this part of the program is correct.

5. RESULTS AND DISCUSSION

We now present the outcome of the computer experiments described in Section 3. The parameters for each run are listed in Table II.

For small amplitudes and short correlation times of the acceleration field Bourret's and Kraichnan's equation both reduce to the Fokker–Planck form, as can be shown analytically. This is, of course, what one should expect since in this parameter range the Fokker–Planck equation is known to give an adequate description. Figure 4 corresponds to this situation. We computed the average propagators $g(v,t;v')$ and $g(0,v,t;x',v')$ numerically and inserted them into the right- and left-hand sides of equations (42)–(44), choosing $v_L = 1.99$, $v_R = 2.01$, $v' = 2.00$ (see Table II for the remaining parameters). Then we plotted the results as a function of time. All four curves stay close together. This means that the equations are fulfilled, and we may assume that all three models describe the evolution of the propagator correctly. The curves go to zero at $t = 0$ because we integrated Eqs. (42)–(44) over the velocity interval $[v_L, v_R]$. We have $g(v, t = 0; v') = \delta(v - v')$, $v' \in [v_L, v_R]$, and so the integrated propagator will remain constant in time until it attains nonzero values outside the interval $[v_L, v_R]$.

Table II. Simulation Parameters[a]

Run	v'	v_L	v_R	b_0	l_c/v'	τ_c	$\epsilon \times 10^3$	Number of realizations
2	2.0	1.99	2.01	8.2×10^{-3}	0.25	0.2	0.17	200 000
3	2.0	1.99	2.01	8.2×10^{-3}	0.25	0.2	0.17	200 000
4	2.0	1.99	2.01	8.2×10^{-3}	0.25	0.2	0.17	300 000
5	2.0	1.99	2.01	8.2×10^{-3}	0.25	1.0	0.30	300 000
6	2.0	1.99	2.01	3.5×10^{-2}	0.25	0.2	0.70	150 000
7	2.0	1.99	2.01	8.2×10^{-3}	0.625	0.2	0.23	300 000
8a	2.0	1.99	2.01	3.5×10^{-2}	0.05	0.1	0.21	300 000
8b	2.0	1.99	2.01	3.5×10^{-2}	0.1	0.1	0.32	150 000

[a] The number of each run corresponds to the number of the figure in which its output is displayed.

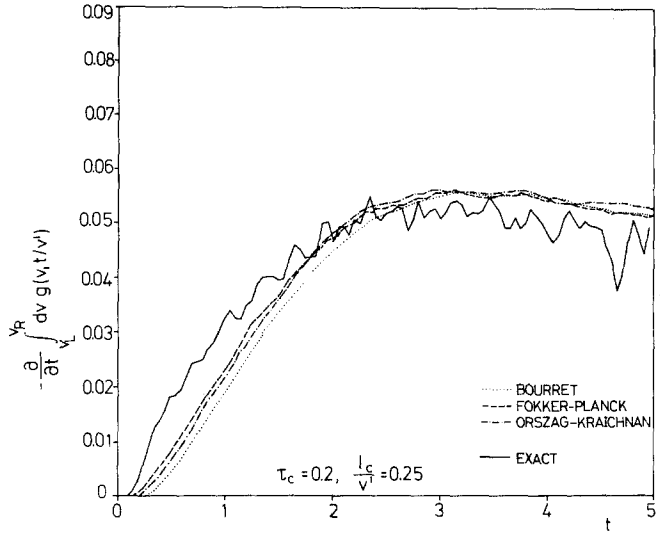

Fig. 4. Right- and left-hand sides of Eqs. (42)–(44) for $b_0 = 8.2 \times 10^{-3}$, $l_c/v' = 0.25$, $\tau_c = 0.2$.

The crucial question now is what will happen when we increase the amplitude or correlation time (respectively, length) of the acceleration field: If Bourret's or Kraichnan's equation is indeed valid beyond the range of validity of the Fokker–Planck theory then after this change of parameters the time evolution of the right- and left-hand sides of (43) or (44) should still agree while the two sides of (42) should not. The results for this crucial test are displayed in Figs. 5 and 6. In Fig. 5 the correlation time is increased by a factor of 5 over the value it had in Fig. 4 while the amplitude and correlation length are unchanged.

In Fig. 6 the amplitude is 4.4 times larger than in Fig. 4, and the correlation time and length are the same. In both cases there is very little difference between the three curves representing the right-hand sides of (42), (43), and (44), and all of them are uniformly bad approximations to the left-hand side to which they claim to be equal. Thus we are led to the conclusion that, depending on the parameter range, either all or none of the three theories are good models for the stochastic acceleration problem. A Fokker–Planck equation, however, is much more easily solved than the integral equations given by Bourret and Kraichnan, and, therefore, the latter two theories are of no use in connection with the stochastic acceleration problem.

For completeness we also show in Fig. 7 the results of a computation in which the correlation length is increased by a factor of 2.5 while the

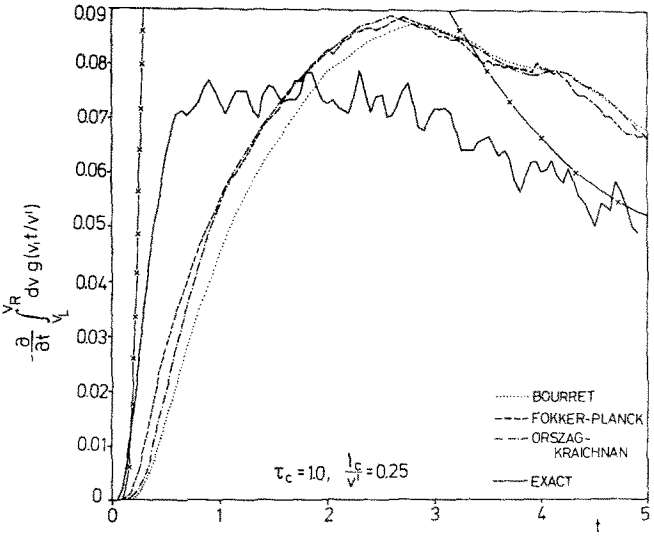

Fig. 5. Right- and left-hand sides of Eqs. (42)–(44) for $b_0 = 8.2 \times 10^{-3}$, $l_c/v' = 0.25$, $\tau_c = 1.0$. The curve through the crosses is formula (63).

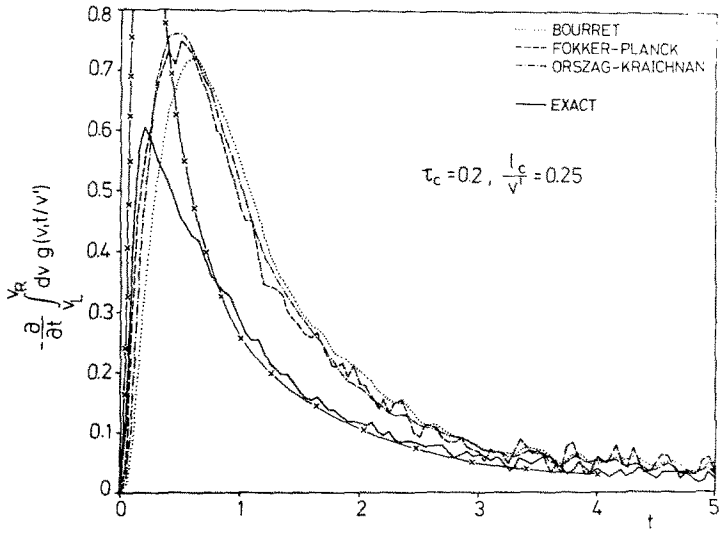

Fig. 6. Right- and left-hand sides of Eqs. (42)–(44) for $b_0 = 3.5 \times 10^{-2}$, $l_c/v' = 0.25$, $\tau_c = 0.2$. The curve through the crosses is formula (63).

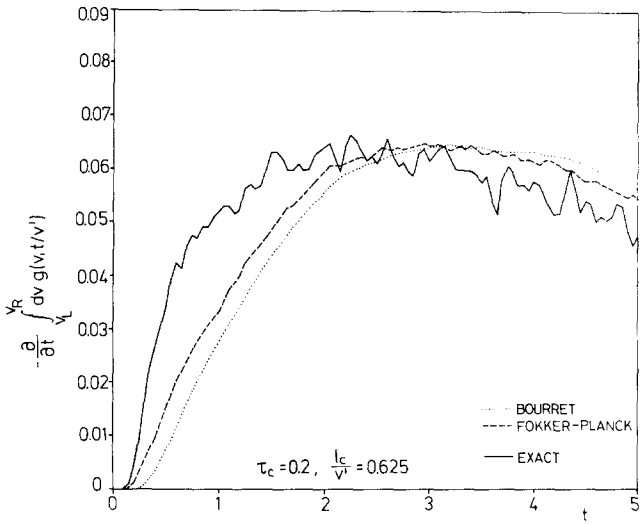

Fig. 7. Right- and left-hand sides of Eqs. (42) and (43) for $b_0 = 8.2 \times 10^{-3}$, $l_c/v' = 0.625$, $\tau_c = 0.2$.

other parameters are again as for the Fig. 4 run. We restricted ourselves to the Bourret and Fokker–Planck cases. Again the two theories do not differ very much.

Finally we repeated the high-amplitude run, depicted in Fig. 6, with shorter correlation times. This served as a test of consistency for our results. Since the system simulated in our computations contains no absolute scales the Fokker–Planck theory must be valid also for increased amplitudes, provided we choose the correlation time sufficiently small. This is indeed how our numerical system behaves, as can be seen in Fig. 8.

For a quantitative interpretation of our results we compare the values of the relative coherent change ϵ of the particle velocity. We define

$$\epsilon = \frac{\langle (\delta v)^2 \rangle^{1/2}}{v'} \tag{55}$$

$$\delta v = \int_0^{T_c} dt\, b(X(t), t) \tag{56}$$

$$T_c = \frac{1}{v'/l_c + 1/\tau_c} \tag{57}$$

and use the unperturbed orbit $X(t) = X(0) + v't$ in the expression for δv.

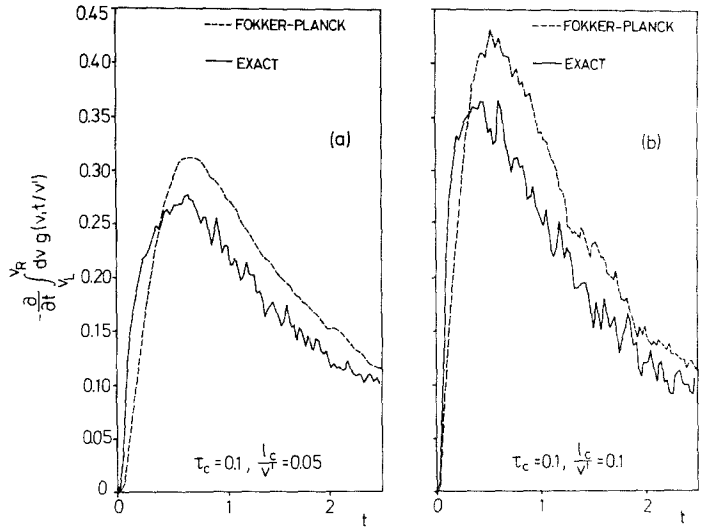

Fig. 8. Right- and left-hand sides of Eqs. (42) for (a) $b_0 = 3.5 \times 10^{-2}$, $l_c/v' = 0.05$, $\tau_c = 0.1$, and (b) $b_0 = 3.5 \times 10^{-2}$, $l_c/v' = 0.1$, $\tau_c = 0.1$.

This gives

$$\epsilon = \left(\frac{2}{e}\right)^{1/2} \frac{b_0}{v'(v'/l_c + 1/\tau_c)} \tag{58}$$

with the correlation function given by (48). The value of ϵ for each run is listed in Table II. Although by intuition one would assume that ϵ is small enough for the Fokker–Planck approximation to be valid throughout all runs the simulation results turned out different. We also notice that ϵ is not the only parameter characterizing the strength of the stochastic acceleration: Although its value is comparable for runs 5 and 8b the Fokker–Planck approximation is not equally well satisfied for these two cases.

At very small times (i.e., $t \ll \tau_c, l_c/v'$) the finite correlation time and length of $b(x,t)$ are not yet felt by the particle. We, therefore, expect it to behave as if being accelerated by a field constant in time and space, i.e., $b(x,t) \approx b(0,0)$ during this stage. Making this assumption enables us to compute $g(v,t;v')$ analytically. The particle velocity is given by

$$V(t) = b(0,0)t + v' \tag{59}$$

and from the distribution of $b(0,0)$ the one of $V(t)$ can be found. We have (cf. Appendix)

$$b(0,0) = b_x(0)b_t(0) \tag{60}$$

where $b_x(0)$ and $b_t(0)$ both are normally distributed. We write

$$f_{b_x}(b) = \frac{1}{(2\pi\sigma_x^2)^{1/2}} \exp\left(-\frac{1}{2}\frac{b^2}{\sigma_x^2}\right) \tag{61a}$$

$$f_{b_t}(b) = \frac{1}{(2\pi\sigma_t^2)^{1/2}} \exp\left(-\frac{1}{2}\frac{b^2}{\sigma_t^2}\right) \tag{61b}$$

for the probability densities of $b_x(0)$ and $b_t(0)$ and find for $b(0,0)$

$$f_b(b) = \frac{1}{\pi b_0} K_0\left(\frac{|b|}{b_0}\right) \tag{62}$$

Here K_0 is the modified Bessel function of the second kind and order zero, and $b_0 = \sigma_x\sigma_y$. It follows that the propagator is given by

$$g(v,t;v') = \frac{1}{\pi b_0 t} K_0\left(\frac{|v-v'|}{b_0 t}\right) \tag{63}$$

This result is plotted in Figures 5 and 6. As we expected, the constant-field approximation adequately describes the initial phase of the simulations. A completely unforeseen feature, however, is its excellent agreement with the numerical results in the middle and final stages of the interaction. As yet we have not found a satisfactory theoretical explanation for this surprising phenomenon.

Finally we want to compare our result $\epsilon \lesssim 0.1\%$ with values of ϵ as obtained from physical experiments. Assuming that the loss of plasma energy is governed by the loss of electron energy due to electrostatic fluctuations with low frequency ($\tau_c \to \infty$) we have from Eq. (58)

$$\epsilon \approx b_0 l_c / v'^2 = e\tilde{E} l_c / \kappa T_e \tag{64}$$

where $\tilde{E}$ is the fluctuating electric field, and T_e the mean electron temperature. If Bohm diffusion prevails we have the following formula for the diffusion coefficient[16]:

$$D_B \equiv \frac{1}{2} l_c v_t = \frac{1}{16}\frac{c\kappa T_e}{eB} \tag{65}$$

where

$$v_t = c\tilde{E}/B \tag{66}$$

is the drift velocity in a magnetized plasma. For Bohm diffusion we obtain then

$$\epsilon_B = 1/8 \tag{67}$$

which is certainly out of the regime where any of the above theories is applicable. But many Tokamak experiments succeeded in lowering the

plasma diffusion by one or two orders of magnitude, and one may hope that the Fokker–Planck regime can be reached in the near future.

APPENDIX

We briefly describe a method to compute an acceleration field $b(x,t)$ which fulfills

$$\langle b(x,t)\rangle = 0$$

$$\langle b(x,t)b(x',t')\rangle = b_0^2 \exp\left(-\frac{|x-x'|}{l_c}\right)\left(-\frac{|t-t'|}{\tau_c}\right)$$

We define

$$b_0 := \xi_0$$

$$b_n := \xi_0\left(1-\frac{\Delta x}{l_c}\right)^n + \sum_{i=1}^{n} \xi_i (\Delta x)^{1/2}\left(1-\frac{\Delta x}{l_c}\right)^{n-i}$$

where ξ_0 and the ξ_i are mutually independent Gaussian random variables. We assume

$$\langle \xi_i\rangle = 0$$

$$\langle \xi_i \xi_j\rangle = \delta_{ij}\sigma^2$$

for $i, j \geqslant 1$, and

$$\langle \xi_0\rangle = 0$$

$\langle \xi_0^2\rangle$ will be defined below. It is obvious from the definition of the ξ_i that

$$\langle b_n\rangle = 0, \qquad n \geqslant 0$$

Furthermore we have for $m < n$

$$\begin{aligned}\langle b_m b_n\rangle &= \langle \xi_0^2\rangle\left(1-\frac{\Delta x}{l_c}\right)^{n+m} + \sum_{i=1}^{m} \sigma^2 \Delta x\left(1-\frac{\Delta x}{l_c}\right)^{n+m-2i} \\ &= \langle \xi_0^2\rangle\left(1-\frac{\Delta x}{l_c}\right)^{n+m} + \left(1-\frac{\Delta x}{l_c}\right)^{n-m}\sigma^2\Delta x \\ &\quad \times \frac{1-(1-\Delta x/l_c)^{2m}}{1-(1-\Delta x/l_c)^2} \\ &= \left(1-\frac{\Delta x}{l_c}\right)^{n-m}\left[\langle \xi_0^2\rangle\left(1-\frac{\Delta x}{l_c}\right)^{2m} + \sigma^2\Delta x\,\frac{1-(1-\Delta x/l_c)^{2m}}{1-(1-\Delta x/l_c)^2}\right]\end{aligned}$$

Now we choose

$$\langle \xi_0^2\rangle = \frac{\sigma^2\Delta x}{1-(1-\Delta x/l_c)^2}$$

and obtain

$$\langle b_m b_n \rangle = \frac{\sigma^2}{2/l_c - \Delta x/l_c^2} \left(1 - \frac{\Delta x}{l_c}\right)^{n-m}$$

For $m > n$ we find analogously

$$\langle b_m b_n \rangle = \frac{\sigma^2}{2/l_c - \Delta x/l_c^2} \left(1 - \frac{\Delta x}{l_c}\right)^{m-n}$$

or, with no restriction on m and n,

$$\langle b_m b_n \rangle = \frac{\sigma^2}{2/l_c - \Delta x/l_c^2} \left(1 - \frac{\Delta x}{l_c}\right)^{|m-n|}$$

Now let $\Delta x \to 0$ and $m, n \to \infty$ in such a way that $m\Delta x = x = \text{const}$, $n\Delta x = x' = \text{const}$. This gives

$$\langle b(x) b(x') \rangle = \frac{\sigma^2 l_c}{2} \exp\left(-\frac{|x - x'|}{l_c}\right)$$

A time-dependent field $b(t)$ can be constructed in a completely analogous fashion. Defining

$$b(x,t) := b(x)b(t)$$

will produce the desired acceleration field.

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